

3D Origami Axioms

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Abstract

The Huzita-Justin axioms provide a wonderful set of prescriptions for making origami constructions, extending the familiar Euclidean ruler and compass methods in the plane. Here we consider a generalization of these axioms to three and higher dimensions; in these dimensions we fold together affine subspaces (using reflections in hyperplanes).

In 3D origami we may fold together a given set of points, lines, planes using the *axioms*; these axioms are described in terms of *basic folds*. The axioms are sets of basic folds which yield only finitely many possible solutions. A basic fold denoted $X \mid Y$ has its elements X, Y among the points (P), lines (L) and planes (π). The solutions to the basic fold is the set of planes for which a reflection in any one of those planes, say ρ , makes $\rho(X)$ and Y incident.

Here are the simplest basic folds.

- 1) $P_1 \mid P_2$; Solution: perpendicular bisector plane.
- 2) $L_1 \mid L_2$; Solution: When L_1, L_2 lie in a plane π we get angle bisector planes perpendicular to π .
- 3) $\pi_1 \mid \pi_2$; Solution: Angle bisector plane and its perpendicular plane through the intersection of π_1 and π_2 .

These three basic folds are in fact axioms. In general when we describe a basic fold, elements with different indices are distinct. The basic folds are listed below, labelled a)-f). The axioms will be numbered.

Here is some standard terminology: B_P , the *bundle* of all planes passing through the point P is 2-dimensional; F_L is the *fan* of all planes containing the line L is 1-dimensional; O_L is the *orthofan* of all planes perpendicular to L is also 1-dimensional; O_π is the set of all planes orthogonal to π is 2-dimensional.

- a) $L \mid L$; this is any plane in F_L or O_L .
- b) $P \mid P$; any plane in B_P

- c) $\pi \mid \pi$; this is either the plane π or any plane in O_π .

Because of the dimension an axiom involving a) needs to specify one more basic fold, while those that involve b) or c) require two more basic folds.

The next set of basic folds have loci solutions in general, that is they are tangent to a surface.

- d) $L \mid \pi$; i) If L, π meet at P then we get a cone of planes reflecting L to π . ii) If L, π are parallel then the solution planes form the tangent planes to a parabolic cylinder, with focus on L and directrix in π , both lying on a plane perpendicular to π .
- e) $P \mid L$; the locus is a parabolic cylinder, since we are reflecting across the tangent to the parabola in the plane of P, L with focus P and directrix L .
- f) $P \mid \pi$; the solutions are the tangent planes to the 3D analogue of parabola, an elliptic paraboloid, a surface of degree 2.

In the presentation we will describe all the axioms that give allowed constructions for this origami geometry. For an axiom we want generic basic folds meaning that the possible left sides are distinct and the corresponding right sides are distinct.

The axioms are most easily enumerated in classes. The first class has axioms with three fixed elements, for example

4) $P_1 \mid P_1, P_2 \mid P_2, P_3 \mid P_3$; this is the plane in $B_{P_1} \cap B_{P_2} \cap B_{P_3}$.

Next comes the axioms involving two fixed elements, for example,

5) $L_1 \mid L_1, P_1 \mid P_1$; a plane containing L_1, P_1 or perpendicular to L_1 and containing P_1 .

And then axioms with one fixed element, for example,

6) $P_1 \mid P_1, P_2 \mid L_1$; a plane passing through P_1 so that the reflection of P_2 lands on L_1 .

The last class contains axioms where no basic folds have a fixed element, for example

7) $P_1 \mid \pi_1, P_2 \mid \pi_2, P_3 \mid \pi_3$; In general since three degree 2 surfaces will meet in 8 common points there are at most 8 possible solutions.